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LETTER TO THE EDITOR

Numerical demonstration of the Berry–Robnik level spacing distribution

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Abstract. We offer a clear numerical demonstration of the Berry–Robnik level spacing distribution in a dynamical regime in which regular and irregular regions coexist in classical phase space. In order to achieve this we had to go very deeply into the semiclassical limit, which, so far, has only been possible for an abstract dynamical system, namely the quantized standard map on a torus. We have performed an extensive numerical analysis of the quasi-energy spectra and of the eigenstates in the two-dimensional phase-space representation. We have confirmed the validity of Percival's conjecture that the eigenstates can be clearly classified either as regular or irregular where the small set of mixed-type states vanishes extremely slowly as we approach the far semiclassical limit. It has been verified that the assumptions (statistical independence) implicit in the Berry–Robnik theory are indeed satisfied giving rise to the observed excellent agreement between theory and experiment. The same high quality agreement is also observed in our comparison of the semiclassical theoretical (Seligman and Verbaarschot) and numerical delta statistics.

In the development of quantum chaos improvements in understanding have been achieved by studying the statistical properties of the (quasi-)energy spectra (and of other observables) in quantum systems whose classical counterparts are non-integrable and chaotic. For recent reviews see Giannoni *et al* (1991), Gutzwiller's book (1990), Eckhardt (1988), Bohigas and Giannoni (1984) and Robnik (1994), and references therein. We know that there are three universality classes of spectral fluctuations: Poisson statistics in the classically integrable cases; in the case of classical ergodicity we find the GOE/GUE statistics of random matrix theories depending on whether there is one/no anti-unitary symmetry (we ignore spin). The interesting and difficult case of mixed-type classical dynamics of KAM-like (generic) systems has been studied numerically for the first time by Robnik (1984), where a continuous transition from Poisson to GOE statistics in a billiard system (Robnik 1983) has been found—this work has been substantially revised in Prosen and Robnik (1993). Further theoretical progress was published by Berry and Robnik (1984) where the following semiclassical theory of the level spacings was presented. The eigenstates (their Wigner functions in phase space) are supposed to condense uniformly on the underlying classical invariant regions such that each of them—in the semiclassical limit—supports a level sequence which for itself has Poisson or GOE statistics if the region is regular or irregular, respectively. All the regular regions can be thought of as supporting a single Poisson sequence because the

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Poisson statistics are preserved upon a statistically independent superposition. The mean level spacing of such a sequence is determined by the fractional phase-space volume of the regular regions. On the other hand, each chaotic (GOE) level sequence has a mean level spacing governed by the corresponding fractional phase-space volume. The entire spectrum is then assumed to be a statistically independent superposition of all subsequences. The statistical independence in the semiclassical limit is justified by the principle of uniform semiclassical condensation of eigenstates (in the phase space) and by their lack of mutual overlap, which is consist with Percival's (1973) conjecture. Thus the problem of the statistics of the entire spectrum is now mathematically precisely formulated and its solution as far as the level spacings are concerned, can be expressed in the following way: the statistical independence of superposition implies factorization of the gap distribution functions (Mehta 1991, Haake 1991): the probability that there is no level within a gap clearly factorizes upon a statistically independent superposition. The connection between the level spacing distribution $P(S)$ and the gap distribution $E(S)$ is as follows:

$$P(S) = \frac{\mathrm{d}^2 E(S)}{\mathrm{d}S^2} \tag{1}$$

and conversely

$$E(S) = \int_S^\infty \mathrm{d}\sigma (\sigma - S) P(\sigma) \, . \tag{2}$$

Leaving aside the general case of infinitely many chaotic components, which does not include anything surprisingly new, let us restrict ourselves to the case of one regular component with mean level density ρ_1 (= fractional phase-space volume) and one chaotic component with the mean level density ρ_2, where $\rho_1 + \rho_2 = 1$. This is already going to be an excellent approximation, because in a generic system of mixed type there is usually only one large and dominating chaotic region. Following Mehta (1991), Haake (1991) and Berry and Robnik (1984), we have

$$E(S) = E_{\mathrm{Poisson}}(\rho_1 S) E_{\mathrm{GOE}}(\rho_2 S) \tag{3}$$

where the Poissonian gap distribution E_{Poisson} is

$$E_{\mathrm{Poisson}}(S) = \exp(-S) \tag{4}$$

whereas for the E_{GOE} there is no simple closed formula (for the infinitely dimensional GOE case) and it must be worked out by using practical approximations for P_{GOE} and/or E_{GOE} which, for example, can be found in Haake (1991), pp 72–74. However, the two-dimensional GOE case (the so-called Wigner surmise) can be worked out explicitly as given in Berry and Robnik (1984), equation (28), which is usually a good starting approximation.

As for the delta statistics $\Delta(L)$ a similar procedure based on the assumption of statistical independence leads to the simple (additive) formula (Seligman and Verbaarschot 1985)

$$\Delta(L) = \Delta_{\mathrm{Poisson}}(\rho_1 L) + \Delta_{\mathrm{GOE}}(\rho_2 L) \tag{5}$$

where $\Delta_{\mathrm{Poisson}}(L) = L/15$ whilst for Δ_{GOE} there are good approximations given in Bohigas (1991).

Let us now define our dynamical system whose phase space is just a compact two-dimensional torus $T_2 = \{(x, y); x, y \in [-\pi, \pi)\}$, where the periodic coordinates x and y will be called position and momentum, respectively. The system's dynamics will simply be given by consecutive applications of the 'free motions' $U_{\mathrm{free}}(x, y) = (x + y, y)$ and 'kicks' $U_{\mathrm{kick}}(x, y) = (x, y - a\sin(x))$. The most useful is the symmetric representation of the evolution mapping U,

$$U = U_{\mathrm{kick}}^{1/2} \circ U_{\mathrm{free}} \circ U_{\mathrm{kick}}^{1/2} \tag{6}$$

where $U_{\rm kick}^{1/2}(x,y) = (x, y - a\sin(x)/2)$. Our mapping (6) is, in fact, the standard (Chirikov) map on a torus T_2 rather than on a cylinder and its representation is dynamically (canonically) equivalent to the usual representation $U_{\rm kick} \circ U_{\rm free}$. It possesses two symmetries, namely the time reversal symmetry $T(x,y) = (x,-y)$, $T \circ U \circ T = U^{-1}$, and parity $P(x,y) = (-x,-y)$, $P \circ U \circ P = U$.

Since the classical phase space is compact, the quantum Hilbert space is finite dimensional and its dimension n determines the dimensionless value of the effective Planck's constant $\hbar_{\rm eff} = 2\pi/n$. Let n be an even number $n = 2m$. The position and momentum eigenstates denoted by $|x_k\rangle$ and $|y_l\rangle$ can be defined through the relation $\langle x_k|y_l\rangle = n^{-1/2}\exp((in/2\pi)x_k y_l)$, where our choice $x_k = (2\pi/n)(k-\frac{1}{2})$, $y_l = (2\pi/n)(l-1)$, $k,l = 1\ldots n$, warrants the single-valuedness on the torus T_2. The quantization procedure is now almost obvious: the quantum unitary evolution propagator $\widehat{U}$ is decomposed into products of free motions $\widehat{U}_{\rm free}$ and kicks $\widehat{U}_{\rm kick}$ in precisely the same way as the classical one (6) where quantum analogues for the kick and the free motion are diagonal in position and momentum representation, respectively:

$$\widehat{U}_{\rm kick} = \sum_k \exp\left(\frac{in}{2\pi}a\cos(x_k)\right)|x_k\rangle\langle x_k| \qquad \widehat{U}_{\rm free} = \sum_l \exp\left(-\frac{in}{2\pi}\frac{y_l^2}{2}\right)|y_l\rangle\langle y_l|. \tag{7}$$

The phases of the diagonal elements in (7) are (when divided by $n/2\pi$) just the classical generating functions which generate the classical mapping (6). Therefore as $n \to \infty$ the quantum evolution approaches the classical dynamics. There exists a simple closed-form expression for the propagator in the position representation,

$$\langle x_k|\widehat{U}|x_{k'}\rangle = \frac{1}{\sqrt{n}}\exp\left[\frac{in}{2\pi}\left(\tfrac{1}{2}(x_k - x_{k'})^2 + \tfrac{1}{2}a\cos(x_k) + \tfrac{1}{2}a\cos(x_{k'})\right)\right] \tag{8}$$

which is the discrete time analogue of the well known infinitesimal propagator $\exp[(i/\hbar)((x-x')^2/2m\,dt - (V(x)+V(x'))\,dt/2)]$ for the general continuous Hamiltonian case. Using the symmetry under parity P one can further reduce the n-dimensional unitary matrix $U_{kl} = \langle x_k|\widehat{U}|x_{k'}\rangle$ into two ($m = n/2$)-dimensional unitary matrices $U_{kl}^{\sigma} = \langle x_k\sigma|\widehat{U}|x_{k'}\sigma\rangle$, where $|x_k\sigma\rangle$ are parity preserving position eigenstates $|x_k\sigma\rangle = 2^{-1/2}(|x_k\rangle + \sigma|-x_k\rangle)$, $k = 1\ldots m$ and $\sigma = \pm 1$ is a parity eigenvalue. Of course, quantization can also be worked out for odd values of n but it is physically less transparent so we have only used even values of n in our numerical example.

We have diagonalized symmetric (due to time reversal) and unitary matrices U_{kl}^{σ} as close to the semiclassical limit $m \to \infty$ as possible. Spectra for both parities and several consecutive values of m were joined together in order to obtain statistically significant results. Clearly, due to the time-reversal symmetry the GOE (or, strictly speaking COE) statistics will apply to irregular level sequences. This is a high-quality resolution test of the Berry–Robnik formula so we have investigated the cumulative level-spacing distribution $W(S) = \int_0^S d\sigma\, P(\sigma)$ rather than the probability distribution $P(S)$ itself, since the latter suffers from arbitrariness of binning. We have applied a least-squares fit of the two-component Berry–Robnik formula with estimated one-sigma uncertainties of the numerical data $\delta W = \sqrt{W(1-W)/N}$, where N is the total number of numerical spacings, (see Prosen and Robnik 1993), and evaluated the χ^2 test. Moreover, we had to use the true ∞-dimensional GOE statistics to model the chaotic spectral subsequence instead of the commonly used Wigner surmise, since as close to the semiclassical limit as we were able to go ($m = 8000$) we could clearly detect considerable differences. On the other hand, we have also compared our data with the phenomenological Brody model of power-law level repulsion:

$$P^{\mathrm{B}}(S,\beta) = aS^{\beta}\exp(-bS^{\beta+1}) \qquad a = (\beta+1)b \qquad b = \left[\Gamma(1+(\beta+1)^{-1})\right]^{\beta+1} \tag{9}$$

which was reported by many authors (Wintgen and Friedrich 1987, Hönig and Wintgen 1989, Prosen and Robnik 1993, Ganesan and Lakshmanan 1994) to provide statistically significant fits to physical data at practically accessible energies (i.e. effective values of $\hbar$). For a more refined analysis we have used the so-called U-representation of the level spacing distribution (see Prosen and Robnik 1993) $U(W(S)) = (2/\pi)\arccos(\sqrt{1-W(S)})$ which has a nice property that the estimated statistical error $\delta U = 1/\pi\sqrt{N}$ is constant. We have plotted $U(W(S)) - U(W_2^{\mathrm{BR}(\infty)}(S,\rho_1))$, where $W_2^{\mathrm{BR}(\infty)}$ is the best fitting two-component Berry–Robnik cumulative level spacing distribution (based on ∞-dimensional GOE statistics), versus $W(S)$ since the density of equally weighted numerical points is constant along the abscissa so that the information is uniformly distributed over the graph.

We have observed a very slow convergence towards the semiclassical limit, which is characterized by a smooth transition from a power-law Brody-like regime in the near semiclassical limit towards the ultimate Berry–Robnik regime in the far semiclassical limit, as illustrated in figure 1 for our system (8) at $a = 1.8$. The quasi-universal Brody-like regime (with the fractional power-law level repulsion) has been clearly and statistically significantly demonstrated in Prosen and Robnik (1993), where the near semiclassical limit was studied. The origin of this phenomenon has been explained and understood theoretically in a separate letter (Prosen and Robnik 1994a). Thus the transition exemplified in figure 1 is very typical.

But for $a = 1.8$ and $m = 8000$ (numerical data were collected for $m = 7991\ldots8000$ and both parities) we have clearly reached the Berry–Robnik regime of the far semiclassical limit which is reflected in the fact that the matching between numerical values and the best-fit Berry–Robnik curve becomes excellent (100% confidence level, see figure 2). The value of the parameter $\rho_1 = 0.272(1\pm0.9\%)$ deviates only by 2.5% from the classical regular volume $\rho_1^{\mathrm{cl}} = 0.265(1\pm0.8\%)$. For larger values of ρ_1 closer to integrability (smaller values of parameter a) the convergence is even slower, e.g. for $a = 1.3$ ($\rho_1 = 0.372$) there are still tiny but detectable deviations between the theory and numerical values even at $m = 8000$. As the value of m decreases the deviation between the quantal and classical value of ρ_1 increases, where the former is typically larger than the latter. But over the whole range $1 < m < 8000$ (from the near to far semiclassical limit) the modified two parameter (ρ_1,β) Berry–Robnik model, where the chaotic subsequence was assumed to have Brody statistics (9) with level repulsion parameter β, has turned out to be highly satisfactory (100% confidence level). We have also found a significant fit to the semiclassical ansatz for delta statistics $\Delta(L)$ (5) (at $m = 8000$) with the best-fit value of parameter $\rho_1 = 0.274(1\pm1.5\%)$ deviating by 3.3% from the classical value (figure 3). The fit was on the interval $0 \leqslant L \leqslant 100$ which is—as judged *a posteriori*—safely below the saturation region (Berry 1985).

This letter reports on the first successful verification of the Berry–Robnik level spacing distribution, which eliminates any doubts about the validity of the Berry–Robnik regime in the dispute concerning the ultimate semiclassical spectral statistics (in generic Hamiltonian systems). A detailed analysis and theoretical explanation of the various regimes of spectral statistics will be given in a separate paper (Prosen and Robnik 1994a). With the present day supercomputer capabilities such a statistically significant analysis is possible only for one-dimensional time-dependent systems, such as our compactified quantum standard map, because the so-called far semiclassical limit is formed very slowly as $\hbar \to 0$ and the effective $\hbar$ is related to the dimension n of matrices which need to be diagonalized via $\hbar \propto n^{-1/f}$, where f is the number of freedoms. Correspondingly, we have investigated the

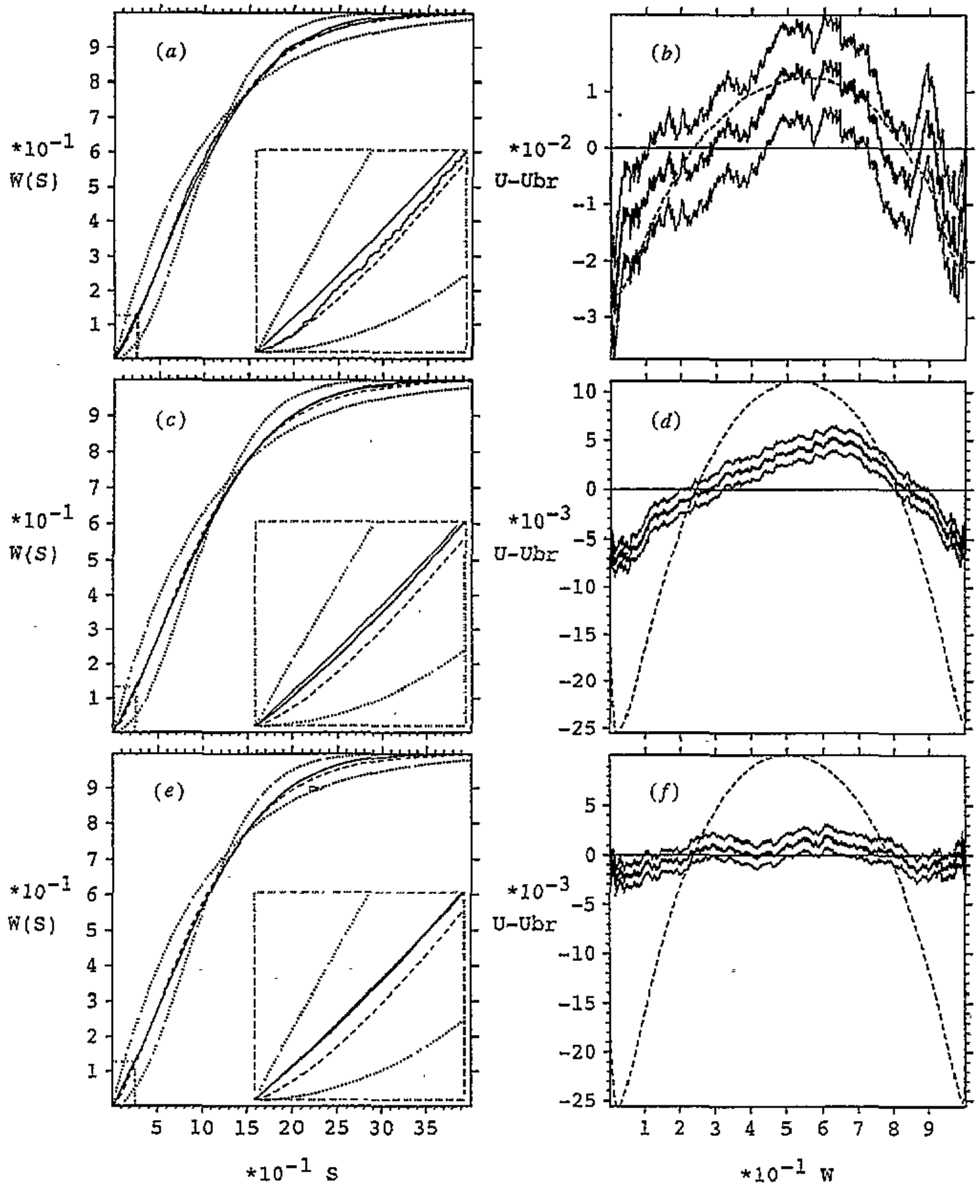

Figure 1. The figure illustrates the transition from the Brody-like to the Berry–Robnik regime for system (8) at $a = 1.8$, at three stretches of m: $11\ldots40$ (1220 spacings) (a), (b), $301\ldots400$ (70 100 spacings) (c), (d), and $3991\ldots4000$ (79 910 spacings) (e), (f) and both parities. The cumulative level spacing distribution $W(S)$ (a), (c), (e) and the deviation of the U-function from the best-fit Berry–Robnik curve $U(W(S)) - U(W_2^{\mathrm{BR}(\infty)}(S, \rho_1))$ versus $W(S)$ (b), (d), (f) is shown. The dotted curves are the limiting Poisson and GOE case, the broken curve is the best-fit Brody distribution, the thin full curve is the best-fit Berry–Robnik curve and the thickest full curve is the numerical one. Small spacing regions $0 < S < 0.25$ are shown in magnified windows. On the right-hand side (b), (d), (f) the Berry–Robnik curve is just the abscissa and the thin full lines indicate $\pm1\sigma$ statistical uncertainty of numerical data.

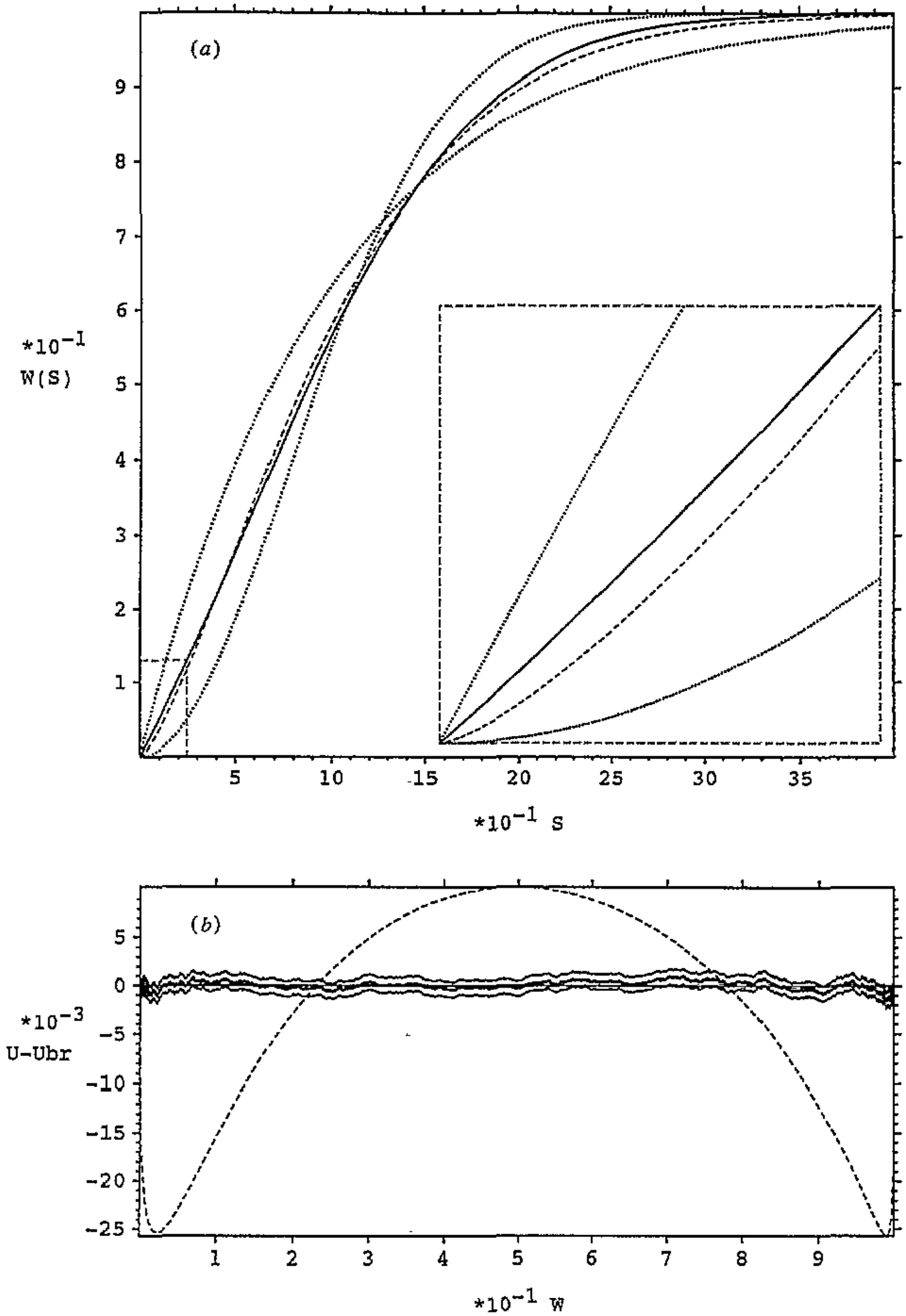

Figure 2. Cumulative level spacing distribution (*a*) and its *U*-function (*b*) for the highest-lying range of $m = 7991\ldots8000$ (159 910 spacings). The meaning of the curves is the same as in figure 1 and also $a = 1.8$. In the standard representation (*a*) the theoretical and numerical curves are completely overlapping, and the agreement between the best-fit Berry–Robnik curve with $\rho_1 = 0.272$ and numerical values is really excellent since the value of $\chi^2 = 45\,000$ is 3.5 times smaller than the number of spacings. The broken curve is again the best-fit Brody curve.

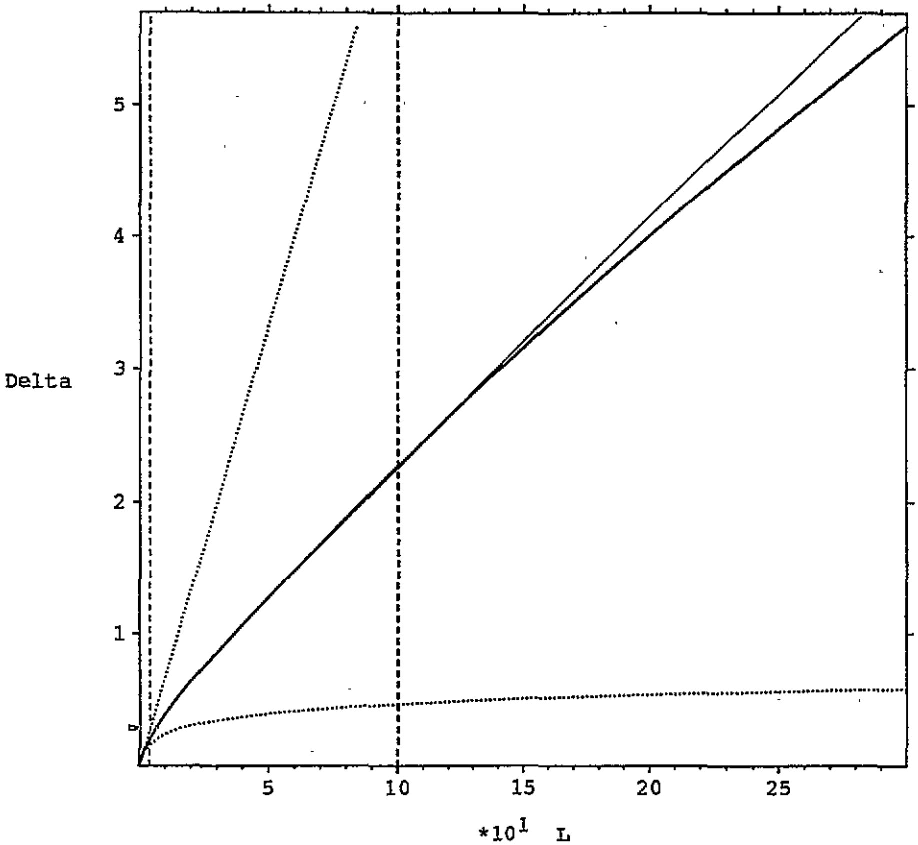

Figure 3. The delta statistics $\Delta(L)$ are shown at $a = 1.8$ for the highest lying range of $m = 7991 \ldots 8000$. The dotted curves are the limiting Poisson and GOE curves, the thin full curve is the best-fit Seligman–Verbaarschot formula (5) with $\rho_1 = 0.274$, and the thick full curve shows the numerical data. The vertical broken curves indicate the region where the least-squares fit is applied. The theory starts to deviate from numerics above $L \approx 150$ where the saturation effects set in.

structure of eigenstates in phase space (Husimi distributions) and we have also found very slow convergence to the ultimate uniform localization on classical invariant components in the semiclassical limit (Prosen and Robnik 1994b). Non-uniform localization on the chaotic region survives much higher in n than one would expect if it was just a consequence of slow classical transport in phase space due to partial barriers (Bohigas *et al* 1993).

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